

# VECTOR-VALUED HILBERT TRANSFORMS ALONG CURVES

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**ABSTRACT.** In this paper, we show that Hilbert transforms along some curves are bounded on  $L^p(\mathbb{R}^n; X)$  for some  $1 < p < \infty$  and some UMD spaces  $X$ . In particular, we prove that Hilbert transforms along some curves are completely  $L^p$ -bounded in the terminology from operator space theory. Moreover, we obtain the  $L^p(\mathbb{R}^n; X)$ -boundedness of anisotropic singular integrals by using the "method of rotations" of Calderón-Zygmund. All these results extend already existing related ones.

## 1. INTRODUCTION

The question of whether the mapping properties of singular integral operators could be extended to the Lebesgue-Böchner spaces  $L^p(\mathbb{R}^n; X)$  ( $1 < p < \infty$ ) of vector-valued functions was taken up by several authors in the 60's. In [1], Benedek, Calderón and Panzone observed that the boundedness on  $L^{p_0}(\mathbb{R}^n; X)$  for one  $1 < p_0 < \infty$  of a singular integral operator, together with Hörmander's condition, implies its boundedness on  $L^p(\mathbb{R}^n; X)$  for all  $1 < p < \infty$ . However, to actually get the  $L^{p_0}(\mathbb{R}^n; X)$ -boundedness (something that was immediate for  $p_0 = 2$  in the scalar-valued), turned out to be a significantly difficult task except in the case  $X = L^{p_0}(\Omega)$  for some measure space  $\Omega$ .

The first progress made in this direction is Burkholder's extension [3] of Riesz's classical theorem on the  $L^p$ -boundedness of the Hilbert transform, where it was shown that if the underlying Banach space  $X$  satisfies the so called UMD-property, then the Hilbert transform is bounded on  $L^p(\mathbb{R}; X)$  for any  $1 < p < \infty$ . Moreover, the UMD-property was shown by Bourgain [2] to be necessary for the boundedness of the Hilbert transform. It is well-known that the Hilbert transform is a prototype of singular integral operators and Fourier multipliers, its boundedness motivates McConnell's [17] and Zimmermann's [28] results on vector-valued Marcinkiewicz-Mihlin multipliers, and Hytönen and Weis's [12] results on vector-valued singular convolution integrals.

Particularly, if  $X$  equals  $S_p$ —the Schatten class, the  $L^p(\mathbb{R}^n; S_p)$ -boundedness is called complete  $L^p$ -boundedness in the light of noncommutative harmonic analysis. In this setting, the complete  $L^2$ -boundedness is immediately available because  $S_2$  is a Hilbert space, and the Fourier transform (or almost orthogonality principle) can be adapted. In order to obtain the complete  $L^p$ -boundedness, so far as

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we know in the noncommutative harmonic analysis, there are only two ways. One way is to establish firstly the weak type  $(1, 1)$  estimate, and then to use interpolation and the duality argument. In this way, the convolution kernel need to satisfy the Lipschitz regularity in order to conduct the pseudo-localization principle as done in [21] (see also [10] for related results). The other way is to get  $(L^\infty, BMO)$  (the noncommutative BMO space) estimate, then to use interpolation and the duality argument. In this case, the kernel is required to satisfy the Hörmander's condition as done in [18] and [15]. However, to get the complete  $L^p$ -boundedness is not a trivial work when the kernel does not satisfy the Lipschitz regularity and the Hörmander condition, see e.g. [9] for more information.

The purpose of our project is to extend the vector-valued singular integrals theory to more general setting. We consider vector-valued singular Radon transforms, which are given by the following principal-valued integral

$$\mathcal{T}f(x) = \text{p.v.} \int_{\mathbb{R}^k} f(x - \Gamma(t))K(t)dt, \quad f \in C_0^\infty(\mathbb{R}^n) \otimes X,$$

where  $X$  is a Banach space,  $K$  is a Calderón-Zygmund kernel in  $\mathbb{R}^k$  and  $\Gamma : \mathbb{R}^k \rightarrow \mathbb{R}^n$  is a surface in  $\mathbb{R}^n$  with  $\Gamma(0) = 0$ ,  $n \geq 2$ . Precisely, we are interested in the boundedness of  $\mathcal{T}$  on  $L^p(\mathbb{R}^n; X)$ , where  $p \in (1, \infty)$  and  $X$  is some Banach space. Obviously,  $\mathcal{T}$  are classical vector-valued singular convolution integrals if  $k = n$  and  $\Gamma(t) = (t_1, t_2, \dots, t_n)$ , and related results have been introduced in the previous paragraphs. On the other hand, if  $X = \mathbb{R}$ ,  $\mathcal{T}$  are classical singular integrals associated to surfaces, which have been well-studied by Stein, Nagel, Wainger, Christ and so on, see [27] for a survey of results through 1978 and [6] through 1999.

In the present paper, we start with the investigation of Hilbert transforms along curves in the hope of providing the insight and inspiration for subsequent development of this subject, as the role played by the classical Hilbert transform in the classical vector-valued Calderón-Zygmund theory. Vector-valued Hilbert transforms along curves are defined by

$$\mathcal{H}f(x) = \text{p.v.} \int_{\mathbb{R}} f(x - \Gamma(t)) \frac{dt}{t}, \quad f \in C_0^\infty(\mathbb{R}^n) \otimes X.$$

In the scalar-valued case, the  $L^2$ -boundedness goes back the work [7] of Fabes who proved it with  $\Gamma(t) = (t^\alpha, t^\beta)$  using complex integration. Then Stein and Wainger [26] obtained the  $L^2$ -boundedness for all homogeneous curves by using Van der Corput's estimates for trigonometric integrals. The first breakthrough was the proof of the  $L^p$ -boundedness in the papers of Nagel, Rivière and Wainger [19] as well as the paper of Nagel and Wainger [20] using Stein's complex interpolation. Since then, many related results have been obtained, see Stein and Wainger's survey paper [27] for the curves having some curvature at the origin, the paper of Carlsson *et al* [5] and the references therein for the flat curves in  $\mathbb{R}^2$ . However, all results about vector-valued singular integrals mentioned previously can not be directly applied to Hilbert transforms along curves on  $L^p(\mathbb{R}^n; X)$ , because they are no longer Calderón-Zygmund operators. Therefore this study is a move beyond the vector-valued Calderón-Zygmund theory.

In the present paper, we extend Nagel, Rivière and Wainger as well as Nagel and Wainger's results mentioned above to the vector-valued setting by combining their original arguments and some idea developed recently by Hytönen and Weis [14] in the vector-valued Calderón-Zygmund theory. To state our results, we need to recall and introduce some notations. Denote by  $\epsilon_j, j \in \mathbb{Z}$ , the Rademacher system of independent random variables on a probability space  $(\Omega, \Sigma, \mathbf{P})$  verifying  $\mathbf{P}(\epsilon_j = 1) = \mathbf{P}(\epsilon_j = -1) = 1/2$ . Let  $\mathbb{E} = \int(\cdot)d\mathbf{P}$  be the corresponding expectation. The main Banach space geometry property of  $X$  we are concerned in this paper is the UMD property (see e.g. [3]), i.e. the following inequality holds:

$$(\mathbb{E} \left\| \sum_{k=1}^N \epsilon_k d_k \right\|_X^2)^{1/2} \leq C (\mathbb{E} \left\| \sum_{k=1}^N d_k \right\|_X^2)^{1/2}$$

for all  $N \in \mathbb{N}$ , all fixed signs  $\epsilon_k \in \{-1, 1\}$ , all  $X$ -valued martingale differences  $(d_k)_{k \geq 0}$ . The following notation is very useful for formulating the main results in this paper.

**Definition 1.1.** Let  $(a, b) \subseteq (0, 1)$ . We define  $\mathcal{I}_{(a,b)}$  to be the set consisting of UMD spaces with its element  $X$  having the form  $X = [H, Y]_\theta$  such that  $\theta \in (a, b)$ ,  $H$  is a Hilbert space and  $Y$  is another UMD space.  $\mathcal{I}_{(0,1)}$  is denoted by  $\mathcal{I}$  for simplicity.

*Remark 1.2.* (i). It is easy to check that all the noncommutative  $L_p$  spaces (containing commutative  $L^p$  spaces) with  $1 < p < \infty$  belong to the class  $\mathcal{I}_{(1-\frac{2}{p}, 1)}$ . From the reflexivity of UMD space, in general we have  $X \in \mathcal{I}_{(a,b)}$  if and only if  $X^* \in \mathcal{I}_{(a,b)}$ . Furthermore, if  $(a, b) \subseteq (c, d) \subseteq (0, 1)$ , then  $\mathcal{I}_{(a,b)} \subseteq \mathcal{I}_{(c,d)}$ .

(ii). In [23], Rubio de Francia proved that for any UMD lattice  $X$  there exist  $\theta \in (0, 1)$ , a Hilbert space  $H$  and another UMD lattice  $Y$  such that  $X = [H, Y]_\theta$ . That means every UMD lattice  $X$  belongs to  $\mathcal{I}$ . In the same paper, the author also ask the open question “Is every  $B \in \text{UMD}$  intermediate between a ‘worse’  $B_0$  and a Hilbert spaces ?” which in our language means “If  $\mathcal{I}$  contains all UMD spaces?”.

The first result is on the Hilbert transform along the homogeneous curves  $\Gamma(t) = (|t|^{\alpha_1} \text{sgnt}, |t|^{\alpha_2} \text{sgnt}, \dots, |t|^{\alpha_n} \text{sgnt})$  with each  $\alpha_i > 0$ .

**Theorem 1.3.** *Let  $X \in \mathcal{I}$  and  $1 < p < \infty$ . Then there exists an absolute constant  $C_p$  such that*

$$\|\mathcal{H}f\|_{L^p(X)} \leq C_p \|f\|_{L^p(X)}, \quad f \in L^p(\mathbb{R}^n; X).$$

This is a vector-valued version of Theorem 1 of Nagel, Rivière and Wainger in [19]. Following the previous remark, Theorem 1.3 implies the complete boundedness of Hilbert transforms along this kind of curves which is of independent interest in the operator space theory. This result also partially generalize the previous result by Rubio de Francia, Ruiz and Torra [22] where they obtained Theorem 1.3 in the case  $X = \ell^q$  with  $1 < q < \infty$ . In [22], the authors used indirectly Benedek, Calderón and Panzone's strategy mentioned previously. While the proof of Theorem 1.3 is motivated by the recent development in the vector-valued Calderón-Zygmund theory [12], see Section 2 for related details.

Let  $\delta_t$  be a one parameter group of dilations and  $\mathbf{e}, \mathbf{f}$  be vectors in  $\mathbb{R}^n$ . A curve  $\Gamma(t)$  is called two-sided homogeneous if the following two conditions hold:

$$\Gamma(t) = \begin{cases} \delta_t \mathbf{e}, & t > 0, \\ \delta_{-t} \mathbf{f}, & t < 0, \\ 0, & t = 0; \end{cases} \quad (1.1)$$

$$\{\xi | \xi \cdot \Gamma(t) \equiv 0, t > 0\} = \{\xi | \xi \cdot \Gamma(t) \equiv 0, t < 0\}.$$

The curve  $\Gamma(t) = (|t|^{\alpha_1} \text{sgn} t, |t|^{\alpha_2} \text{sgn} t, \dots, |t|^{\alpha_n} \text{sgn} t)$  is a model with  $\delta_t x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \dots, t^{\alpha_n} x_n)$ ,  $\mathbf{e} = \mathbf{1}$  and  $\mathbf{f} = -\mathbf{1}$ . We will see that the same argument for this particular curve works for all the curves with the same dilation but  $\mathbf{e} = -\mathbf{f}$ . Generalization of Theorem 1.3 to all two-sided homogeneous curves in turn motivates us to consider the vector-valued Calderón-Zygmund theory associated to one parameter group of dilations, which is a project under progress.

As an application, Theorem 1.3 is used to deal with vector-valued anisotropic singular integrals with homogeneous kernel by Calderón-Zygmund's rotation method. This work improves Hytönen's Theorem 5.2 in [11] in some sense, see Section 3 for more details.

In the next result, we deal with certain convex curves in  $\mathbb{R}^2$  with the form  $\Gamma(t) = (t, \gamma(t))$ ,  $\gamma(t)$  is some convex function for  $t \geq 0$ .

**Theorem 1.4.** *Let  $X$  be an UMD lattice belonging to the class  $I_{(0, \frac{1}{5})}$ ,  $\gamma(t)$  be a continuous odd function, twice continuously differentiable, increasing and convex for  $t \geq 0$ . Suppose also that  $\gamma''$  is monotone for  $t > 0$  and there exists  $C > 0$  so that  $\gamma'(t) \leq Ct\gamma''(t)$  for  $t > 0$ . Then for  $\frac{5}{3} < p < \frac{5}{2}$ , there exists an absolute constant  $C_p$  such that*

$$\|\mathcal{H}f\|_{L^p(X)} \leq C_p \|f\|_{L^p(X)}, \quad f \in L^p(\mathbb{R}^n; X).$$

A large class of functions  $\gamma(t)$  satisfy the conditions in Theorem 1.4, such as

$$\gamma(t) = \text{sgn}(t)|t|^\alpha, \quad (\alpha \geq 2) \quad \text{and} \quad \gamma(t) = te^{-1/|t|}.$$

The first one is homogeneous, while another one does not have any homogeneity. This result is a vector-valued extension of Theorem 3.1 of Nagel and Wainger in [20]. Theorem 1.4 also generalizes the second author's result [16] in the case  $X = \ell^q$  with  $5/3 < q < 5/2$ . The proof of Theorem 1.4 is again motivated by the recent development of the vector-valued Calderón-Zygmund theory [14]. In fact, in Section 4, we prove a more general version, i.e. Theorem 1.4 is also true if  $X$  satisfies the following weaker condition: there exist  $\theta \in (0, \frac{1}{5})$ , Hilbert space  $H$  and UMD space  $Y$  with property  $(\alpha)$  (recalled in Section 4) such that  $X = [H, Y]_\theta$ .

## 2. PROOF OF THEOREM 1.3

The main arguments in this section are from [27], we will repeat some results for completeness. Before the proof, we need some notations. Let matrix  $A = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ , then  $\Gamma'(t) = A\Gamma(t)/t$  for  $t > 0$ . We also define a norm

function  $\rho(x)$  by the unique positive solution of

$$\sum_{i=1}^n x_i^2 \rho^{-2\alpha_i} = 1$$

and  $\rho(0) = 0$ . This definition was introduced in the pioneering work on anisotropic singular integrals of Fabes [7]. Obviously,  $\rho(\delta_t x) = t\rho(x)$  for  $t > 0$ ,  $\rho(x) = 1$  if and only if the Euclidean norm  $|x| = 1$  which means  $x$  is on the unit sphere  $\mathbf{S}^{n-1}$ . See also Proposition 1-9 in [27] for more properties of  $\rho$ . By a change of variables, we assume  $\alpha_1 = 1$  and  $\alpha_i \geq 1$  for  $2 \leq i \leq n$ , and set  $\Delta = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ . Without loss of generality, we assume that  $\alpha_i \neq \alpha_j$  when  $i \neq j$ , then  $\Gamma(t)$  does not lie in a proper subspace of  $\mathbb{R}^n$ . If not,  $\Gamma$  lies in some proper subspace, then the argument of Stein and Wainger in [27, pp.1262] implies our desired result.

For  $z \in \mathbb{C}$ , we define an analytic family of operators  $\mathcal{H}_z$  by

$$\widehat{\mathcal{H}_z f}(\xi) = \{\rho(\xi)\}^z m_z(\xi) \hat{f}(\xi),$$

where  $m_z$  are given by

$$m_z(\xi) = \text{p.v.} \int_{\mathbb{R}} e^{-2\pi i \xi \cdot \Gamma(t)} |t|^z \frac{dt}{t}.$$

Obviously,  $\mathcal{H}_0$  is our original operator  $\mathcal{H}$ .

As in [27], the desired result will be concluded by analytic interpolation once we show the following two estimates: For Hilbert space  $H$

$$\|\mathcal{H}_z f\|_{L^2(\mathbb{R}^n; H)} \leq C(z) \|f\|_{L^2(\mathbb{R}^n; H)}, \quad (2.1)$$

where  $-1 < \text{Re}(z) \leq \sigma$  for some  $\sigma > 0$  and  $C(z)$  grows at most polynomially in  $|z|$ , and for UMD space  $Y$

$$\|\mathcal{H}_z f\|_{L^p(\mathbb{R}^n; Y)} \leq C(z, p) \|f\|_{L^p(\mathbb{R}^n; Y)}, \quad 1 < p < \infty, \quad (2.2)$$

where  $-\beta \leq \text{Re}(z) \leq -\eta$  for arbitrarily positive  $\eta$  and some positive  $\beta$  as well as  $C(z, p)$  grows at most as fast as a polynomial in  $|z|$  for fixed  $\eta$ .

Indeed, we obtain Theorem 1.3 by performing twice the analytic interpolation argument in [25] as follows. Let  $T_z f(x) = e^{z^2} \mathcal{H}_z f(x)$ . Note that  $|e^{z^2}| = e^{\text{Re}(z)^2 - \text{Im}(z)^2}$ , then by (2.1) there exists a constant  $M_0$  which is independent of  $\text{Im}(z)$  such that

$$\|T_z f\|_{L^2(\mathbb{R}^n; H)} \leq C(z) e^{-\text{Im}(z)^2} \|f\|_{L^2(\mathbb{R}^n; H)} \leq M_0 \|f\|_{L^2(\mathbb{R}^n; H)} \quad (2.3)$$

when  $-1 < \text{Re}(z) < \sigma$ . Also, for any UMD space  $Y$  and  $q \in (1, \infty)$ , by (2.2) there exists a constant  $M_1$  which is independent of  $\text{Im}(z)$  such that

$$\|T_z f\|_{L^q(\mathbb{R}^n; Y)} \leq M_1 \|f\|_{L^q(\mathbb{R}^n; Y)} \quad \text{when} \quad -\beta < \text{Re}(z) < 0. \quad (2.4)$$

Obviously, this inequality holds in particular with  $Y = H$ .

For  $1 < p < \infty$ , we choose  $\theta_1 \in (0, 1)$ ,  $\sigma_1 < 0$ ,  $0 < \sigma_0 < \sigma$  and  $q_1 \in (1, \infty)$  such that

$$\sigma_0(1 - \theta_1) + \sigma_1 \theta_1 =: \sigma_2 > 0, \quad \frac{1}{p} = \frac{1 - \theta_1}{2} + \frac{\theta_1}{q_1}.$$

Interpolating between (2.3) and (2.4) with  $Y = H$ , we have

$$\|T_z f\|_{L^p(\mathbb{R}^n; H)} \leq C(p, z) \|f\|_{L^p(\mathbb{R}^n; H)} \quad \text{when } \operatorname{Re}(z) = \sigma_2 > 0. \quad (2.5)$$

Note that  $X = [H, Y]_\theta$  for some Hilbert space  $H$ , UMD space  $Y$  and  $\theta \in (0, 1)$ . For fixed  $\theta$ , we choose  $\sigma_3 < 0$  such that

$$0 = (1 - \theta)\sigma_2 + \theta\sigma_3.$$

In the same way, interpolating between (2.5) and (2.4) with  $q = p$ , we obtain

$$\|\mathcal{H}f\|_{L^p(\mathbb{R}^n; X)} = \|T_0 f\|_{L^p(\mathbb{R}^n; X)} \leq C \|f\|_{L^p(\mathbb{R}^n; X)}.$$

The estimate (2.1) is trivial since Plancherel's theorem remains true for Hilbert space valued functions and the original arguments for Lemma 4.2 in [27] work here. The novelty of the proof lies in the proof of (2.2). In the case  $Y = \ell^q$  with  $1 < q < \infty$ , it has been proved in (2.2) in [22] by Benedek, Calderón and Panzone's argument since  $L^q(\ell^q)$ -boundedness is trivial. For general UMD space, we shall follow Hytönen and Weis's idea [14] established recently to prove the  $L^p(Y)$  estimates simultaneously for all  $1 < p < \infty$ . The following subsection is devoted to the proof of estimate (2.2).

**2.1. The proof of (2.2).** The following proof is essentially the same as [11], we include it here for the sake of completeness. From Lemma 4.4 of [27], we can write that

$$\mathcal{H}_z f(x) = K_z * f(x),$$

where

$$K_z(x) = \int_{\mathbb{R}} h_z(x - \Gamma(t)) |t|^z \frac{dt}{t} \quad \text{and} \quad \hat{h}_z(\xi) = \{\rho(\xi)\}^z.$$

It is known that  $h_z$  is a locally integrable function,  $C^\infty$  away from the origin satisfying

$$h_z(\delta_\lambda x) = \lambda^{-\Delta - z} h_z(x), \quad \lambda > 0, \quad x \neq 0.$$

Moreover, each derivative of  $h_z(x)$  is bounded by a polynomial in  $|z|$ , if  $\rho(x) = 1$ . In particular,  $K_z$  has the homogeneity property  $\lambda^\Delta K_z(\delta_\lambda x) = K_z(x)$ .

Let  $\hat{\mathcal{D}}_0(\mathbb{R}^n) = \{\psi \in \mathcal{S}(\mathbb{R}^n) \mid \hat{\psi} \in \mathcal{D}(\mathbb{R}^n), 0 \notin \operatorname{supp} \hat{\psi}\}$ . Let  $\eta \in \mathcal{D}(\mathbb{R}^n)$  have range  $[0, 1]$ , vanish for  $\rho(\xi) \geq 2$  and equal 1 for  $\rho(\xi) \leq 1$ . For  $j \in \mathbb{Z}$ , we define  $\hat{\phi}_0(\xi) = \eta(\xi) - \eta(\delta_2 \xi)$ ,  $\hat{\phi}_j(\xi) = \hat{\phi}_0(\delta_{2^{-j}} \xi)$  and  $\hat{\chi}_j(\xi) = \hat{\phi}_{j-1}(\xi) + \hat{\phi}_j(\xi) + \hat{\phi}_{j+1}(\xi)$ . Then  $\hat{\phi}_j(\xi)$  is supported in the annulus  $\{2^{j-1} \leq \rho(\xi) \leq 2^{j+1}\}$ , and

$$\sum_j \hat{\phi}_j(\xi) = 1 \quad \text{for } \xi \neq 0. \quad (2.6)$$

Moreover, since  $\hat{\chi}_j$  equals 1 on the support of  $\hat{\phi}_j$ , we have

$$\phi_j = \phi_j * \chi_j * \chi_j. \quad (2.7)$$

The estimate (2.2) will be deduced from the following key estimate which will be shown in the next subsection.

**Proposition 2.1.** *Let  $\phi_0$  and  $K_z$  be defined as above. We have*

$$\int_{\mathbb{R}^n} |\phi_0 * K_z(x)| \log^n(e + \rho(x)) dx \leq C(z).$$

With above preparations at hand, we finish the proof of the estimate (2.2).

*Proof.* For fixed  $z$ , we denote  $K_z$  by  $K$  for simplicity. Given  $f \in \hat{\mathcal{D}}_0(\mathbb{R}^n) \otimes Y$ ,  $g \in \hat{\mathcal{D}}_0(\mathbb{R}^n) \otimes Y^*$ , by (2.6) and (2.7), we have

$$\langle g, K * f \rangle = \langle \tilde{K} * g, f \rangle = \sum_j \langle \phi_j * \tilde{K} * (\chi_j * g), \chi_j * f \rangle,$$

where the summation is finite and  $\tilde{K}(x) = K(-x)$ . Changing variable and using the fact  $\lambda^\Delta K_z(\delta_\lambda x) = K_z(x)$ ,

$$(\phi_j * \tilde{K}) * (\chi_j * g)(x) = \int_{\mathbb{R}^n} \phi_0 * \tilde{K}(y) (\chi_j * g)(x - \delta_{2^{-j}} y) dy.$$

Hence, by Hölder's inequality and the Khintchine-Kahane inequality

$$\begin{aligned} |\langle g, K * f \rangle| &= \left| \int_{\mathbb{R}^n} \mathbb{E} \left\langle \sum_j \epsilon_j \chi_j * g(\cdot - \delta_{2^{-j}} y), \sum_i \epsilon_i \phi_0 * K(y) \chi_i * f \right\rangle dy \right| \\ &\leq \int_{\mathbb{R}^n} \mathbb{E} \left\| \sum_j \epsilon_j \chi_j * g(\cdot - \delta_{2^{-j}} y) \right\|_{L^{p'}(Y^*)} \mathbb{E} \left\| \sum_i \epsilon_i \chi_i * f \right\|_{L^p(Y)} |\phi_0 * K(y)| dy. \end{aligned}$$

It is easy to check that  $m = \sum_j \epsilon_j \hat{\chi}_j$  is an anisotropic multiplier. Hence, by Theorem 3 in [11], we have

$$\left\| \sum_j \epsilon_j \chi_j * f \right\|_{L^p(\mathbb{R}^n; Y)} \leq C_{p,X} \|f\|_{L^p(\mathbb{R}^n; Y)}. \quad (2.8)$$

By Proposition 2.1 and (2.8), we shall finish the proof by showing

$$\mathbb{E} \left\| \sum_j \epsilon_j \chi_j * g(\cdot - \delta_{2^{-j}} y) \right\|_{L^{p'}(Y^*)} \leq C \log^n(e + \rho(y)) \mathbb{E} \left\| \sum_j \epsilon_j \chi_j * g \right\|_{L^{p'}(Y^*)}.$$

Let  $e_i$  be the  $i$ -th standard unit vector. Above estimate is just a  $n$ -fold application of

$$\mathbb{E} \left\| \sum_j \epsilon_j \chi_j * g(\cdot - \delta_{2^{-j}} y_i e_i) \right\|_{L^{p'}(Y^*)} \leq C \log(e + \rho(y)) \mathbb{E} \left\| \sum_j \epsilon_j \chi_j * g \right\|_{L^{p'}(Y^*)},$$

which follows from Lemma 10 of Bourgain [2].  $\square$

**2.2. The proof of Proposition 2.1.** The proof of Proposition 2.1 is based on the following two lemmas. The first one states that the kernel  $K_z$  satisfies a weighted Hörmander condition, which will be verified at the end of this subsection.

**Lemma 2.2.** *If  $-\beta \leq \operatorname{Re}(z) \leq -\eta$ , then for sufficiently large constants  $C_0$  and  $C_1(z)$ , we have*

$$\int_{\rho(x) \geq C_0 \rho(y)} |K_z(x - y) - K_z(x)| \log^n(e + \rho(x)) dx \leq C_1(z) \log^n(e + \rho(y)) \quad (2.9)$$

for any  $y \in \mathbb{R}^n \setminus \{0\}$ . Moreover,  $C_1(z)$  grows at most as fast as a polynomial in  $|z|$  for a fixed  $\eta$ .

The second lemma is a kind of decomposition lemma which has been established in Lemma 4.10 of [14]. We reformulate it in our anisotropic case.



**Lemma 2.3.** *Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  with vanishing integral. Then there exists a decomposition  $\varphi = \sum_{m \geq 0} \psi_m$  with the following properties:*

$$\psi_m \in \mathcal{D}(\mathbb{R}^n), \text{ supp } \psi_m \subseteq \{x \mid \rho(x) \leq C2^{\alpha m}\}, \int_{\mathbb{R}^n} \psi_m(y) dy = 0,$$

where  $C$  and  $\alpha$  are two universal constants only depending on the norm  $\rho$  and the dimension  $n$ , and for every  $p \in [1, \infty]$  and every  $M > 0$ , the sequence of Lebesgue norms  $\|\psi_m\|_{L^p}$ , as well as  $\|\hat{\psi}_m\|_{L^p}$ , is  $\mathcal{O}(2^{-mM})$  as  $m \rightarrow \infty$ .

*Proof.* Let us give a quick explanation of this lemma. From Lemma 4.10 of [12],  $\psi_m$  is supported in  $\{x \mid |x| \leq 2^m\}$ . Fix  $x \in \{x \mid |x| \leq 2^m\}$ , by Proposition 1-9 of [27], if  $\rho(x) \geq 1$ , then

$$\rho(x) \leq c_1 |x|^{\alpha_1} \leq c_1 2^{a_1 m}$$

and if  $\rho(x) \leq 1$ , then

$$\rho(x) \leq c_2 |x|^{\alpha_2} \leq c_2 2^{a_2 m}$$

with  $c_1, c_2, a_1, a_2$  positive constants. We obtain the desired result by choosing  $C = \max\{c_1, c_2\}$  and  $\alpha = \max\{a_1, a_2\}$ .  $\square$

*Proof of Proposition 2.1.* The main idea comes from [12], we include most details here for completeness. By Lemma 2.3, we write  $\phi_0 = \sum_{m \geq 0} \psi_m$  with  $\psi_m$ 's satisfying the properties stated in that lemma. Then we decompose  $K_z$  into pieces

$$K_{z,m}(x) = K_z * \psi_m(x)$$

and estimate each of them respectively.

We first estimate the integral outside the larger ellipsoid  $\mathcal{B}_1 = \{x \mid \rho(x) \leq CC_1 2^{\alpha m}\}$  with  $C_1$  fixed later depending on  $C_0$ . Recall that  $\psi_m$  is supported in the ellipsoid  $\mathcal{B}_0 = \{x \mid \rho(x) \leq C2^{\alpha m}\}$  and the integral of  $\psi_m$  vanishes, by Fubini's theorem and Lemma 2.2, we obtain

$$\begin{aligned} & \int_{\mathcal{B}_1^c} |K_{z,m}(x)| \log^n(e + \rho(x)) dx \\ &= \int_{\mathcal{B}_1^c} \left| \int_{\mathcal{B}_0} K_z(x-y) \psi_m(y) dy \right| \log^n(e + \rho(x)) dx \\ &\leq \int_{\mathcal{B}_0} \int_{\rho(x) \geq C_0 \rho(y)} |K_z(x-y) - K_z(x)| \log^n(e + \rho(x)) dx \psi_m(y) dy \\ &\leq C_1(z) \int_{\mathcal{B}_0} \log^n(e + \rho(y)) \psi_m(y) dy \leq C_1(z) \|\psi_m\|_{L^\infty} \int_{\mathcal{B}_0} \log^n(e + \rho(y)) dy. \end{aligned}$$

By Lemma 2.3, the last quantity is of order  $\mathcal{O}(2^{-m})$  as  $m \rightarrow \infty$  since  $\|\psi_m\|_{L^\infty} \leq C_M 2^{-mM}$  for  $M > 0$  while

$$\int_{\mathcal{B}_0} \log^n(e + \rho(y)) dy \leq C 2^{mN}$$

for a fixed  $N$ .



Inside the ellipsoid  $\mathcal{B}_1$ , the computation is easier because of the fact  $\|\hat{K}_z\|_{L^\infty} \leq C(z)$ , then

$$\begin{aligned}
\int_{\mathcal{B}_1} |K_{z,m}(x)| \log^n(e + \rho(x)) dx &\leq \|K_{z,m}\|_{L^\infty} \int_{\mathcal{B}_1} \log^n(e + \rho(x)) dx \\
&\leq \int_{\mathcal{B}_1} \log^n(e + \rho(x)) dx \|\hat{K}_{z,m}\|_{L^1} \\
&= \int_{\mathcal{B}_1} \log^n(e + \rho(x)) dx \int_{\mathbb{R}^n} |\hat{K}_z(\xi) \hat{\psi}_m(\xi)| d\xi \\
&\leq \|\hat{K}_z\|_{L^\infty} \|\hat{\psi}_m\|_{L^1} \int_{\mathcal{B}_1} \log^n(e + \rho(x)) dx \leq C(z) 2^{-m}.
\end{aligned}$$

The last inequality holds due to the same reason that for the case outside the ellipsoid. Finally, we obtain Proposition 2.1 by summing over  $m$ .  $\square$

To complete the proof of Proposition 2.1, we still need to show Lemma 2.2.

*Proof of Lemma 2.2.* We follow the main sketch provided in [27], but improve related estimates. To verify  $K_z$  satisfying (2.9), we may assume that  $\rho(y) = 1$ , it suffices to prove that

$$\int_{\rho(x) \geq C_0} |K_z(x - y) - K_z(x)| \log^n(e + \rho(x)) dx \leq C(z). \quad (2.10)$$

In fact, we set  $\lambda = \rho(y)$  and  $y' = y/\lambda$ . Obviously,  $\rho(y') = 1$ . By a linear transformation  $x = \delta_\lambda x'$  and the homogeneity of  $K_z$ , we have

$$\begin{aligned}
&\int_{\rho(x) \geq C_0 \rho(y)} |K_z(x - y) - K_z(x)| \log^n(e + \rho(x)) dx \\
&= \int_{\rho(x') \geq C_0} |K_z(x' - y') - K_z(x')| \log^n(e + \lambda \rho(x')) dx'.
\end{aligned}$$

If  $\lambda = \rho(y) \geq 6$ , it is trivial that

$$\log(e + \lambda \rho(x')) \leq \log(e + \lambda) + \log(e + \rho(x')) \leq \log(e + \lambda) \log(e + \rho(x')),$$

where we use the assumption that  $C_0 \geq 6$ . Then,

$$\begin{aligned}
&\int_{\rho(x) \geq C_0 \rho(y)} |K_z(x - y) - K_z(x)| \log^n(e + \rho(x)) dx \\
&\leq \int_{\rho(x') \geq C_0} |K_z(x' - y') - K_z(x')| \log^n(e + \rho(x')) dx' \log^n(e + \rho(y)) \\
&\leq C(z) \log^n(e + \rho(y)).
\end{aligned}$$

When  $\lambda = \rho(y) < 6$ , by (2.10), we get

$$\begin{aligned} & \int_{\rho(x) \geq C_0 \rho(y)} |K_z(x-y) - K_z(x)| \log^n(e + \rho(x)) dx \\ & \leq 2^n \int_{\rho(x') \geq C_0} |K_z(x'-y') - K_z(x')| \log^n(e + \rho(x')) dx' \\ & \leq C(z) \leq C(z) \log^n(e + \rho(y)). \end{aligned}$$

To prove (2.10), we define  $K_z^1$  and  $K_z^2$  by

$$K_z^1(x) = \int_{|t| \leq 1} h_z(x - \Gamma(t)) |t|^z \frac{dt}{t} \text{ and } K_z^2(x) = K_z(x) - K_z^1(x),$$

respectively. We split the integral as

$$\begin{aligned} & \int_{\rho(x) \geq C_0} |K_z(x-y) - K_z(x)| \log^n(e + \rho(x)) dx \\ & \leq \int_{\rho(x) \geq C_0} |K_z^1(x)| \log^n(e + \rho(x)) dx \\ & \quad + \int_{\rho(x) \geq C_0} |K_z^1(x-y)| \log^n(e + \rho(x)) dx \\ & \quad + \int_{\rho(x) \geq C_0} |K_z^2(x-y) - K_z^2(x)| \log^n(e + \rho(x)) dx. \end{aligned}$$

To estimate first two summands, we need a estimate related to  $h_z$ , which can be found in [27, pp.1273]. The homogeneity and smoothness of  $h_z$  away from origin imply that

$$|h_z(x-y) - h_z(x)| \leq C(z) \frac{|y|}{\{\rho(x)\}^{\Delta + \operatorname{Re}(z) + \mu}} \quad (2.11)$$

for some  $\mu > 0$ , provide  $|y|/|x|$  is sufficiently small.

We set  $\beta = \min\{\mu, 1\}$ . For the first integral, by using Fubini's theorem and (2.11), we have

$$\begin{aligned} & \int_{\rho(x) \geq C_0} |K_z^1(x)| \log^n(e + \rho(x)) dx \\ & \leq \int_{\rho(x) \geq C_0} \int_{|t| \leq 1} |h_z(x - \Gamma(t)) - h_z(x)| |t|^{Re(z)-1} dt \log^n(e + \rho(x)) dx \\ & \leq \int_{|t| \leq 1} |t|^{Re(z)-1} \int_{\rho(x) \geq C_0} |h_z(x - \Gamma(t)) - h_z(x)| \log^n(e + \rho(x)) dx dt \\ & \leq \int_{|t| \leq 1} |t|^{Re(z)-1} |\Gamma(t)| \int_{\rho(x) \geq C_0} \rho(x)^{-[\Delta + \operatorname{Re}(z) + \mu]} \log^n(e + \rho(x)) dx dt \\ & \leq C(z), \end{aligned}$$

where we use the fact that  $-\beta < \operatorname{Re}(z) < 0$ .

The norm function  $\rho(x)$  have the property of  $\rho(x+y) \leq c(\rho(x) + \rho(y))$  for some  $c > 0$  (see Proposition 1-9 in [27]). Specially, we set  $C_0 \geq \max\{6, 3c\}$ . Note that

$\rho(x-y) \geq \frac{1}{c}\rho(x) - \rho(y) \geq \frac{C_0}{c} - 1 \geq 2$  and  $\rho(x) \leq c[\rho(x-y) + \rho(y)] \leq c\rho(x-y) + c$ . Using a linear transformation, we treat the second summand as the first one,

$$\begin{aligned} & \int_{\rho(x) \geq C_0} |K_z^1(x-y)| \log^n(e + \rho(x)) dx \\ & \leq \int_{\rho(x) \geq 2} |K_z^1(x)| \log^n(e + c + c\rho(x)) dx \leq C(z). \end{aligned}$$

Finally, using Fubini's theorem, we have

$$\begin{aligned} & \int_{\rho(x) \geq C_0} |K_z^2(x-y) - K_z^2(x)| \log^n(e + \rho(x)) dx \\ & \leq \int_{|t| \geq 1} \int_{\rho(x) \geq C_0} |h_z(x-y-\Gamma(t)) - h_z(x-\Gamma(t))| \log^n(e + \rho(x)) \frac{dx dt}{|t|^{1-Re(z)}}. \end{aligned}$$

We divide the inner integral above according to the distance between  $x$  and  $\Gamma(t)$ . Note that  $\rho(y) = 1$ , if  $|y|/|x - \Gamma(t)|$  is sufficient small, that is  $|x - \Gamma(t)|$  is away from the origin, we can get that  $\rho(x - \Gamma(t)) \geq C_2$ , where  $C_2$  is an appropriate constant. In this case, by (2.11) and a linear transformation, we obtain the following estimate

$$\begin{aligned} & \int_{|t| \geq 1} \int_{\substack{\rho(x) \geq C_0 \\ \rho(x-\Gamma(t)) \geq C_2}} |h_z(x-y-\Gamma(t)) - h_z(x-\Gamma(t))| \log^n(e + \rho(x)) \frac{dx dt}{|t|^{1-Re(z)}} \\ & \leq C \int_{|t| \geq 1} \int_{\substack{\rho(x) \geq C_0 \\ \rho(x-\Gamma(t)) \geq C_2}} \frac{|y|}{\{\rho(x-\Gamma(t))\}^{\Delta+\mu+Re(z)}} \log^n(e + \rho(x)) \frac{dx dt}{|t|^{1-Re(z)}} \\ & \leq C \int_{|t| \geq 1} \int_{\rho(x) \geq C_2} \frac{1}{\{\rho(x)\}^{\Delta+\mu+Re(z)}} \log^n(e + c\rho(x) + ct) \frac{dx dt}{|t|^{1-Re(z)}} \\ & \leq C \int_{|t| \geq 1} \int_{\rho(x) \geq C_2} \frac{1}{\{\rho(x)\}^{\Delta+\mu+Re(z)}} \{ \log^n(e + \rho(x)) + \log^n(e + t) \} \frac{dx dt}{|t|^{1-Re(z)}} \\ & \leq C, \end{aligned}$$

where we use the fact that for fixed  $|t| \geq 1$ ,  $\rho(x) \leq c[\rho(x - \Gamma(t)) + \rho(\Gamma(t))] = c[\rho(x - \Gamma(t)) + t]$ .

It is trivial that  $\rho(x + y + \Gamma(t)) \leq c^2[\rho(x) + \rho(y) + \rho(\Gamma(t))] = c^2[1 + \rho(x) + t]$ . Then, the remainder can be controlled by

$$\begin{aligned} & \int_{|t| \geq 1} \int_{\substack{\rho(x) \geq C_0 \\ \rho(x-\Gamma(t)) \leq C_2}} [|h_z(x-y-\Gamma(t))| + |h_z(x-\Gamma(t))|] \log^n(e + \rho(x)) \frac{dx dt}{|t|^{1-Re(z)}} \\ & \leq \int_{|t| \geq 1} \int_{\substack{\rho(x) \geq C_0 \\ \rho(x-\Gamma(t)) \leq C_2}} |h_z(x-y-\Gamma(t))| \log^n(e + \rho(x)) dx |t|^{Re(z)-1} dt \\ & + \int_{|t| \geq 1} \int_{\substack{\rho(x) \geq C_0 \\ \rho(x-\Gamma(t)) \leq C_2}} |h_z(x-\Gamma(t))| \log^n(e + \rho(x)) dx |t|^{Re(z)-1} dt \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{|t| \geq 1} \int_{\rho(x) \leq c(C_2+1)} |h_z(x)| dx |t|^{Re(z)-1} \log^n(e+t) dt \\
&\leq C(z),
\end{aligned}$$

where we use the fact that  $h_z$  is locally integrable.  $\square$

### 3. ANISOTROPIC SINGULAR INTEGRALS

It was shown by Calderón and Zygmund [4] that the  $L^p$ -boundedness of singular integrals with rough kernels can be deduced from the  $L^p$ -boundedness of the (directional) Hilbert transform using the method of rotations. In this section, we show a similar phenomenon happens, that is, the  $L^p(X)$ -boundedness of Hilbert transforms along curve  $\Gamma(t) = (|t|^{\alpha_1} sgn t, |t|^{\alpha_2} sgn t, \dots, |t|^{\alpha_n} sgn t)$  considered in the previous section implies the  $L^p(X)$  boundedness of singular integrals  $T_\Omega$  with kernels of the form  $K(x) = \Omega(x)\rho(x)^{-\Delta}$ , where  $\Omega$  is a function on  $\mathbb{R}^n \setminus \{0\}$  satisfying the homogeneity  $\Omega(\delta_t x) = \Omega(x)$  for all  $t > 0$ , size condition

$$\int_{\mathbf{S}^{n-1}} \sum_{i=1}^n \alpha_i \omega_i^2 |\Omega(\omega)| d\omega < \infty, \quad (3.1)$$

and the cancelation condition

$$\int_{\mathbf{S}^{n-1}} \sum_{i=1}^n \alpha_i \omega_i^2 \Omega(\omega) d\omega = 0,$$

which can be understood from the following change-of-variable formula

$$dx = t^{\Delta-1} \sum_{i=1}^n \alpha_i \omega_i^2 dt d\omega.$$

**Theorem 3.1.** *Let  $X \in \mathcal{I}$ . If  $\Omega$  is odd, then the operators  $T_\Omega$  described previously are bounded on  $L^p(\mathbb{R}^n; X)$  for  $1 < p < \infty$ .*

Guliev [8] has obtained the boundedness of anisotropic singular integrals with scalar valued-kernels on UMD lattices. Recently, Hytönen[11] generalized some work of Guliev to the anisotropic singular integrals with operator-valued kernels acting on UMD space. While their arguments require that  $\Omega(x)$  should satisfy a kind of  $L^\infty$ -Dini condition, which is a much more restricted condition than ours. So, Theorem 3.1 is a generalization of Hytönen and Guliev's result in this sense.

*Proof.* Changing the variables, we find

$$\begin{aligned}
T_\Omega f(x) &= \text{p.v.} \int_{\mathbb{R}^n} f(x - \delta_{\rho(y)} \delta_{\rho(y)}^{-1} y) \Omega(\delta_{\rho(y)}^{-1} y) \{\rho(y)\}^{-\Delta} dy \\
&= \int_0^\infty \int_{\mathbf{S}^{n-1}} f(x - \delta_t \omega) \sum_{i=1}^n \alpha_i \omega_i^2 \Omega(\omega) d\omega \frac{dt}{t}.
\end{aligned} \quad (3.2)$$

Note that  $\Omega$  is odd, by a linear transformation, we also have

$$T_\Omega f(x) = \int_{-\infty}^0 \int_{\mathbf{S}^{n-1}} f(x + \delta_{(-t)} \omega) \sum_{i=1}^n \alpha_i \omega_i^2 \Omega(\omega) d\omega \frac{dt}{t}. \quad (3.3)$$

Using Fubini theorem, and adding (3.2) and (3.3) together, we get

$$T_\Omega f(x) = \frac{1}{2} \int_{\mathbf{S}^{n-1}} \sum_{i=1}^n \alpha_i \omega_i^2 \Omega(\omega) \left[ \int_{-\infty}^0 f(x + \delta_{(-t)}\omega) \frac{dt}{t} + \int_0^\infty f(x - \delta_t\omega) \frac{dt}{t} \right] d\omega.$$

Then, it suffices to prove that

$$\left\| \int_{-\infty}^0 f(x + \delta_{(-t)}\omega) \frac{dt}{t} + \int_0^\infty f(x - \delta_t\omega) \frac{dt}{t} \right\|_{L^p(\mathbb{R}^n; \mathbf{X})} \leq C_p \|f\|_{L^p(\mathbb{R}^n; X)},$$

where the constant  $C_p$  is independent of  $\omega$ .

For fixed  $\omega \in \mathbf{S}^{n-1}$ , define  $\Gamma_\omega(t)$  as the curve in the form of (1.1) associated to the dilation  $\delta_t$  with  $\mathbf{e} = \omega$  and  $\mathbf{f} = -\omega$ , then the quantity inside the norm of the previous inequality is the Hilbert transform along the curve  $\Gamma_\omega(t)$ . The same arguments for the proof of Theorem 1.3 work also for the curve  $\Gamma_\omega(t)$ , and we obtain the desired result.  $\square$

In the classical case (dilation given by  $\delta_t x = tx$ ), it is known that the boundedness of  $T_\Omega$  is also obtained for the even function  $\Omega$  under a stronger size condition  $\Omega \in L \log^+ L(\mathbf{S}^{n-1})$ . The main ingredient is the existence of Riesz transforms  $R_j$ ,  $j = 1, 2, \dots, n$ , such that

- (i)  $-\sum_{j=1}^n R_j \circ R_j = I$ ,
- (ii) the kernel of  $T_\Omega \circ R_j$  is still homogeneous, and the associated  $\Omega_j$  is an odd function satisfying size condition (3.1).

In the anisotropic setting, it seems very difficult to find some replacements for Riesz transforms such that similar properties as (i) and (ii) hold. Hence we leave it as an open problem that whether Theorem 3.1 is still true for the even function  $\Omega$  under a stronger size condition.

#### 4. THE PROOF OF THEOREM 1.4

The main argument for the proof is similar to that for Theorem 1.3. We first introduce a family of analytic operators. For  $z \in \mathbb{C}$ , we define an analytic family of operators  $\mathcal{H}_z$  by

$$\widehat{\mathcal{H}_z f}(\xi, \eta) = m_z(\xi, \eta) \hat{f}(\xi, \eta),$$

where  $m_z$  are given by

$$m_z(\xi, \eta) = \text{p.v.} \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^z \frac{dt}{t}.$$

Obviously,  $\mathcal{H}_0$  is our original operator  $\mathcal{H}$ .

Following the idea in [20], it suffices to prove the following two estimates:

$$\|\mathcal{H}_z f\|_{L^2(\mathbb{R}^2; H)} \leq C_\delta [1 + |Im(z)|] \|f\|_{L^2(\mathbb{R}^2; H)}, \quad (4.1)$$

where  $Re(z) = \frac{1}{4} - \delta$  for some  $\delta > 0$ , and

$$\|\mathcal{H}_z f\|_{L^q(\mathbb{R}^2; Y)} \leq C [1 + |Im(z)|]^2 \|f\|_{L^q(\mathbb{R}^2; Y)}, \quad (4.2)$$

where  $Y$  is an UMD lattice,  $Re(z) < -1$ ,  $1 < q < \infty$ , the constant  $C$  depends on  $Re(z)$  and is independent of  $Im(z)$ .

Indeed, we finish the proof by analytic interpolation argument [25]. Let  $T_z f(x) = e^{z^2} \mathcal{H}_z f(x)$ . Note that  $|e^{z^2}| = e^{\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2}$ , by (4.1) there exists a constant  $M_0$  which is independent of  $\operatorname{Im}(z)$  such that

$$\|T_z f\|_{L^2(\mathbb{R}^2; H)} \leq C_\delta e^{-\operatorname{Im}(z)^2} [1 + |\operatorname{Im}(z)|] \|f\|_{L^2(\mathbb{R}^2; H)} \leq M_0 \|f\|_{L^2(\mathbb{R}^2; H)}$$

when  $\operatorname{Re}(z) = \frac{1}{4} - \delta$ . Also, for UMD lattice  $Y$  and  $q \in (1, \infty)$ , by (4.2) there exists a constant  $M_1$  which is independent of  $\operatorname{Im}(z)$  such that

$$\|T_z f\|_{L^q(\mathbb{R}^2; Y)} \leq M_1 \|f\|_{L^q(\mathbb{R}^2; Y)} \quad \text{when } \operatorname{Re}(z) < -1.$$

This inequality also holds in particular with  $Y = H$ .

For  $\frac{5}{3} < p \leq 2$ , there exist  $1 < q < \infty$  and  $\theta_0 \in (0, \frac{1}{5})$  so that

$$\frac{1}{p} = \frac{1 - \theta_0}{2} + \frac{\theta_0}{q} \quad \text{and} \quad (\frac{1}{4} - \delta)(1 - \theta_0) + (-1 - \varepsilon_0)\theta_0 =: \sigma_1 \in (0, \frac{1}{4})$$

for some  $\varepsilon_0 > 0$  and  $0 < \delta < \frac{1}{4}$ . By interpolation of analytic operators, we have

$$\|T_z f\|_{L^p(\mathbb{R}^2; H)} \leq C(z) \|f\|_{L^p(\mathbb{R}^2; H)} \quad \text{for } \operatorname{Re}(z) = \sigma_1 \in (0, 1/4).$$

Given an UMD lattice  $X \in \mathcal{I}_{(0, 1/5)}$ , there exist a  $\theta \in (0, \frac{1}{5})$ , a Hilbert space  $H$  and another UMD lattice  $Y$ , such that  $L^p(\mathbb{R}^2; X) = [L^p(\mathbb{R}^2; H), L^p(\mathbb{R}^2; Y)]_\theta$ . For such a  $\theta$  and appropriate  $\sigma_1$ , we choose  $\varepsilon_1 > 0$  such that  $(1 - \theta)\sigma_1 + \theta(-1 - \varepsilon_1) = 0$ . Using interpolation of analytic operators once more, we obtain

$$\|\mathcal{H} f\|_{L^p(\mathbb{R}^2; X)} \leq C \|f\|_{L^p(\mathbb{R}^2; X)}$$

for  $\frac{5}{3} < p \leq 2$ . The duality argument implies the result for  $2 \leq p < \frac{5}{2}$ . This completes the proof of Theorem 1.4.

The estimate (4.1) holds since Plancherel's theorem works also for Hilbert space valued functions and the original argument in [20] can be repeated in the present situation. The novelty of the proof lies in the estimate (4.2), for which we need the vector-valued Fourier multiplier theorem established recently.

Let us firstly recall some notations. A Banach space  $X$  satisfies property  $(\alpha)$  if there is a positive constant  $C$  such that

$$\mathbb{E} \mathbb{E}' \left| \sum_{k, l=1}^N \epsilon_k \epsilon'_l \alpha_{kl} x_{kl} \right|_X \leq C \mathbb{E} \mathbb{E}' \left| \sum_{k, l=1}^N \epsilon_k \epsilon'_l x_{kl} \right|_X$$

for all  $N \in \mathbb{N}$ , all vectors  $x_{kl} \in X$  and scalars  $|\alpha_{kl}| \leq 1$  ( $1 \leq k, l \leq N$ ), where  $\epsilon_k$ ,  $k \in \mathbb{Z}$  and  $\epsilon'_l$ ,  $l \in \mathbb{Z}$  are two identical independent sequences.

*Remark 4.1.* The commutative  $L^p$  spaces satisfy property  $(\alpha)$  for all  $1 \leq p < \infty$ . Also, this property is inherited from  $X$  by  $L^p(\mu, X)$  for  $p \in [1, \infty)$ . Every Banach space with a local unconditional structure and finite cotype, in particular every Banach lattice, has property  $(\alpha)$ .

Let  $m : \mathbb{R}^n \rightarrow \mathbb{C}$  be a bounded function, the associated operator  $T_m$  is defined on the test functions  $f \in \mathcal{S}(\mathbb{R}^n) \otimes X$  by

$$T_m f(x) = (m \hat{f})^\vee(x).$$

The sufficiency part of the following vector-valued Fourier multiplier theorem was proved by Štrkalj and Weis [24], while the necessity of those conditions was obtained by Hytönen and Weis [14].

**Lemma 4.2.** *The Marcinkiewicz-Lizorkin condition  $|\xi^\beta| |D^\beta m(\xi)| \leq C$  for all  $\beta \in \{0, 1\}^n$  is sufficient for the  $L^p(\mathbb{R}^n; X)$ -boundedness of  $T_m$ ,  $n > 1$ , if and only if  $X$  is an UMD space with property  $(\alpha)$ .*

In view of Lemma 4.2 and Remark 4.1, to prove the estimate (4.2), it suffices to show that the following functions

$$m_z(\xi, \eta), \xi \frac{\partial m_z}{\partial \xi}(\xi, \eta), \eta \frac{\partial m_z}{\partial \eta}(\xi, \eta), \xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta)$$

are uniformly bounded on  $\mathbb{R}^2$  for  $\operatorname{Re}(z) < -1$ .

The uniform boundedness of  $m_z(\xi, \eta)$  is trivial, it can be showed by minor modification of the proof of (4.1). Without repetition, we omit the proof. The following estimates are essentially proved in [20], we include them here for the sake of completeness.

**The boundedness of  $\xi \frac{\partial m_z}{\partial \xi}(\xi, \eta)$ .** Integration by part implies that

$$\begin{aligned} \xi \frac{\partial m_z}{\partial \xi}(\xi, \eta) &= -2\pi i \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \xi [1 + \eta^2 \gamma^2(t)]^z dt \\ &= \int_{\mathbb{R}} \frac{d}{dt} (e^{-2\pi i \xi t}) e^{-2\pi i \eta \gamma(t)} [1 + \eta^2 \gamma^2(t)]^z dt \\ &= e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^z \Big|_{-\infty}^{\infty} \\ &\quad + 2\pi i \eta \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \gamma'(t) [1 + \eta^2 \gamma^2(t)]^z dt \\ &\quad - 2z \eta^2 \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^{z-1} \gamma(t) \gamma'(t) dt. \end{aligned}$$

Note that  $\operatorname{Re}(z) < -1$ , for  $t \in \mathbb{R}$ , we have  $|[1 + \eta^2 \gamma^2(t)]^z| = [1 + \eta^2 \gamma^2(t)]^{\operatorname{Re}(z)} \leq 1$ . The boundary terms are bounded by 1.

For  $\operatorname{Re}(z) < -1$ , making the change of variables  $u = |\eta| \gamma(t)$ , we obtain

$$\begin{aligned} \left| \eta \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \gamma'(t) [1 + \eta^2 \gamma^2(t)]^z dt \right| &\leq \int_{\mathbb{R}} \gamma'(t) |\eta| [1 + \eta^2 \gamma^2(t)]^{\operatorname{Re}(z)} dt \\ &\leq \int_{\mathbb{R}} (1 + u^2)^{\operatorname{Re}(z)} du \leq \pi. \end{aligned}$$

In a similar way, the second integrated term can be dominated by

$$\begin{aligned} &\left| z \eta^2 \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^{z-1} \gamma(t) \gamma'(t) dt \right| \\ &\leq 2|z| \int_0^\infty [1 + \eta^2 \gamma^2(t)]^{\operatorname{Re}(z)-1} \eta^2 \gamma(t) \gamma'(t) dt \\ &\leq |z| \int_0^\infty (1 + u)^{\operatorname{Re}(z)-1} du \leq 1 + |\operatorname{Im}(z)|. \end{aligned}$$



Therefore, for  $Re(z) < -1$ ,

$$\left| \xi \frac{\partial m_z}{\partial \xi}(\xi, \eta) \right| \leq C[1 + |Im(z)|].$$

**The boundedness of  $\eta \frac{\partial m_z}{\partial \eta}(\xi, \eta)$ .** Integrating by parts, we obtain

$$\begin{aligned} \eta \frac{\partial m_z}{\partial \eta}(\xi, \eta) &= -2\pi i \text{ p.v. } \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta \gamma(t) [1 + \eta^2 \gamma^2(t)]^z \frac{dt}{t} \\ &\quad + 2z \text{ p.v. } \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-1} \frac{dt}{t}. \end{aligned}$$

To estimate above two integrals, we follow the argument used in the proof of (4.1). For the first integral, for any  $\varepsilon > 0$ , it suffices to bound the following two parts

$$\int_{\varepsilon < |t| < t_0} |\eta| |\gamma(t)| [1 + \eta^2 \gamma^2(t)]^{Re(z)} \frac{dt}{|t|} \quad \text{and} \quad \int_{|t| \geq t_0} |\eta| |\gamma(t)| [1 + \eta^2 \gamma^2(t)]^{Re(z)} \frac{dt}{|t|}.$$

Recall that  $t_0 > 0$  was chosen so that  $|\eta| \gamma(t_0) = 1$ , and  $\gamma(t) \leq t \gamma'(t)$  because of the convexity. Thus,

$$\int_{\varepsilon < |t| < t_0} |\eta| |\gamma(t)| [1 + \eta^2 \gamma^2(t)]^{Re(z)} \frac{dt}{|t|} \leq 2|\eta| \int_0^{t_0} \frac{\gamma(t)}{t} dt \leq 2|\eta| \int_0^{t_0} \gamma'(t) dt \leq 2.$$

For  $Re(z) < -1$ , an elementary calculation implies that

$$\int_{|t| \geq t_0} |\eta| |\gamma(t)| [1 + \eta^2 \gamma^2(t)]^{Re(z)} \frac{dt}{|t|} \leq 2|\eta|^{2Re(z)+1} \int_{t_0}^{\infty} \gamma^{2Re(z)}(t) \frac{\gamma(t)}{t} dt \leq 2.$$

Similarly, the second integral can be controlled by

$$\begin{aligned} &\left| z \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-1} \frac{dt}{t} \right| \\ &\leq 2|z| \int_0^{t_0} \eta^2 \gamma^2(t) \frac{dt}{t} + 2|z| \int_{t_0}^{\infty} \eta^2 \gamma^2(t) [\eta^2 \gamma^2(t)]^{Re(z)-1} \frac{dt}{t} \\ &\leq 2|z| \eta^2 \int_0^{t_0} \gamma(t) \gamma'(t) dt + 2|z| \eta^{2Re(z)} \int_{t_0}^{\infty} \gamma^{2Re(z)-1}(t) \gamma'(t) dt \\ &\leq |z| + \frac{|z|}{|Re(z)|} \leq 2|Re(z)| [1 + |Im(z)|]. \end{aligned}$$

Therefore, for  $Re(z) < -1$ ,

$$\left| \xi \frac{\partial m_z}{\partial \xi}(\xi, \eta) \right| \leq C[1 + |Im(z)|].$$

**The boundedness of  $\xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta)$ .** To deal with  $\xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta)$ , we rewrite it as

$$\begin{aligned} \xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta) &= -4\pi^2 \xi \eta \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \gamma(t) [1 + \eta^2 \gamma^2(t)]^z dt \\ &\quad - 4\pi i z \xi \eta \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^{z-1} \eta \gamma^2(t) dt. \end{aligned}$$

For the first term, integrating by parts, we obtain

$$\begin{aligned}
& 4\pi^2 \xi \eta \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \gamma(t) [1 + \eta^2 \gamma^2(t)]^z dt \\
&= 2\pi i \int_{\mathbb{R}} \frac{d}{dt} (e^{-2\pi i \xi t}) e^{-2\pi i \eta \gamma(t)} [\eta \gamma(t)] [1 + \eta^2 \gamma^2(t)]^z dt \\
&= 2\pi i e^{-2\pi i[\xi t + \eta \gamma(t)]} [\eta \gamma(t)] [1 + \eta^2 \gamma^2(t)]^z \Big|_{-\infty}^{\infty} \\
&- 4\pi^2 \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta \gamma'(t) [\eta \gamma(t)] [1 + \eta^2 \gamma^2(t)]^z dt \\
&- 2\pi i \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta \gamma'(t) [1 + \eta^2 \gamma^2(t)]^z dt \\
&- 4\pi i z \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^{z-1} \eta^3 \gamma^2(t) \gamma'(t) dt.
\end{aligned}$$

Obviously, for  $\operatorname{Re}(z) < -1$ ,  $t \in \mathbb{R}$ ,  $|2\pi i e^{-2\pi i[\xi t + \eta \gamma(t)]} [\eta \gamma(t)] [1 + \eta^2 \gamma^2(t)]^z| \leq 2\pi |\eta| |\gamma(t)| [1 + \eta^2 \gamma^2(t)]^{\operatorname{Re}(z)} \leq 2\pi$ . So, the boundary terms are bounded by  $2\pi$ .

For the first integrated term, making the change of variables  $u = \eta^2 \gamma^2(t)$ , we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta \gamma'(t) [\eta \gamma(t)] [1 + \eta^2 \gamma^2(t)]^z dt \right| \\
&\leq 2 \int_0^\infty [1 + \eta^2 \gamma^2(t)]^{\operatorname{Re}(z)} \eta^2 \gamma(t) \gamma'(t) dt \\
&\leq \int_0^\infty (1 + u)^{\operatorname{Re}(z)} du \leq \frac{1}{|\operatorname{Re}(z) + 1|}.
\end{aligned}$$

The second integrated terms can be treated in the same way, let  $u = \eta \gamma(t)$ ,

$$\left| \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta \gamma'(t) [1 + \eta^2 \gamma^2(t)]^z dt \right| \leq \int_{\mathbb{R}} (1 + u^2)^{\operatorname{Re}(z)} du \leq \pi.$$

Similarly, a trivial calculation shows that

$$\begin{aligned}
& \left| z \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^{z-1} \eta^3 \gamma^2(t) \gamma'(t) dt \right| \\
&\leq 2|z| \int_0^\infty u^2 (1 + u^2)^{\operatorname{Re}(z)-1} du \leq \pi |z|.
\end{aligned}$$

The second term can be handled similarly. Integrating by parts, we decompose it as

$$\begin{aligned}
& 4\pi i z \xi \eta \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta \gamma(t)]} [1 + \eta^2 \gamma^2(t)]^{z-1} \eta \gamma^2(t) dt \\
&= 2\pi i z \int_{\mathbb{R}} \frac{d}{dt} (e^{-2\pi i \xi t}) e^{-2\pi i \eta \gamma(t)} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-1} dt \\
&= 2\pi i z e^{-2\pi i[\xi t + \eta \gamma(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-1} \Big|_{-\infty}^{\infty}
\end{aligned}$$

$$\begin{aligned}
& - 4\pi^2 z \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta\gamma(t)]} \eta\gamma'(t) \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-1} dt \\
& - 4\pi i z \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta\gamma(t)]} \eta^2 \gamma(t) \gamma'(t) [1 + \eta^2 \gamma^2(t)]^{z-1} dt \\
& - 4\pi i z(z-1) \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta\gamma(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-2} \eta^2 \gamma(t) \gamma'(t) dt.
\end{aligned}$$

Obviously, for  $\operatorname{Re}(z) < -1$ ,  $t \in \mathbb{R}$ ,  $|ze^{-2\pi i[\xi t + \eta\gamma(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-1}| \leq |z|$ . The boundary terms are dominated by  $4\pi|z|$ .

For the first integrated term, by making the change of variables  $u = \eta\gamma(t)$ , we have the estimate

$$\begin{aligned}
\left| z \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta\gamma(t)]} \eta\gamma'(t) \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{z-1} dt \right| & \leq |z| \int_{\mathbb{R}} u^2 (1 + u^2)^{\operatorname{Re}(z)-1} du \\
& \leq \pi|z|.
\end{aligned}$$

To estimate the second integrated terms, we make the transformation  $u = \eta^2 \gamma^2(t)$  and get

$$\begin{aligned}
\left| z \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta\gamma(t)]} \eta^2 \gamma(t) \gamma'(t) [1 + \eta^2 \gamma^2(t)]^{z-1} dt \right| & \leq |z| \int_0^\infty (1 + u)^{\operatorname{Re}(z)-1} du \\
& \leq \frac{|z|}{|\operatorname{Re}(z)|}.
\end{aligned}$$

Similarly, the third integrated terms can be treated as

$$\begin{aligned}
& \left| z(z-1) \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta\gamma(t)]} [1 + \eta^2 \gamma^2(t)]^{z-2} \eta^4 \gamma^3(t) \gamma'(t) dt \right| \\
& \leq |z(z-1)| \int_0^\infty (1 + u)^{\operatorname{Re}(z)-1} du \leq \frac{|z(z-1)|}{|\operatorname{Re}(z)|}.
\end{aligned}$$

Note that for  $\operatorname{Re}(z) < -1$ , we have the following elementary estimates

$$|z| \leq |\operatorname{Re}(z)|[1 + |\operatorname{Im}(z)|] \quad \text{and} \quad |z-1| \leq |\operatorname{Re}(z)-1|[1 + |\operatorname{Im}(z)|].$$

Finally, combining the above eight estimates, we obtain

$$|\xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta)| \leq C[1 + \operatorname{Im}(z)]^2.$$

This completes the proof of Theorem 1.4.

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